RELATED RATES OF CHANGE WHERE MORE THAN 2 RATES OF CHANGE ARE INVOLVED

We can use the Chain Rule { $\frac{dy}{dx} = \frac{dy}{dr} \times \frac{dr}{dt} \times \frac{dt}{dx}$ } to relate three rates of change.

Example: A cylindrical water tank with a diameter of 5m is being filled with water at a rate of 0.18 m^3/h. After t hours the curved surface area of the water in the tank was A m^2, the height of water in the tank was h metres and the volume was V m^3. Find the rate at which A is increasing when the height is 2.5m.

- Find: $\frac{dA}{dt}$
- Use Chain Rule: $\frac{dA}{dt} = \frac{dA}{} \times \frac{}{dt}$ and then we will need to expand this Chain Rule.
- We have been given $\frac{dV}{dt}$ = 0.18 m^3/h, so the Chain Rule now looks like: $\frac{dA}{dt} = \frac{dA}{dV} \times \frac{dV}{dt}$.
- To find $\frac{dA}{dV}$, if we do not have an expression for V in terms of A or A in terms of V to be able to differentiate, then we can find $\frac{dA}{dV}$ by applying the Chain Rule again: $\frac{dA}{dV} = \frac{dA}{} \times \frac{}{dV}$.
- Both A and V can be expressed in terms of h, so the missing variable in our Chain Rule is dh: $\frac{dA}{dV} = \frac{dA}{dh} \times \frac{dh}{dV}$.
- We will now combine the two chain rules: $\frac{dA}{dt} = \frac{dA}{dV} \times \frac{dV}{dt}$ and $\frac{dA}{dV} = \frac{dA}{dh} \times \frac{dh}{dV}$
- The expression for $\frac{dA}{dt}$ can now be written as: $\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt}$.
- Since A = $2\pi rh$, then $\frac{dA}{dh} = 2\pi r$. Since V = $\pi r^2 h$, $\frac{dV}{dh} = \pi r^2$ and $\frac{dh}{dV} = \frac{1}{\pi r^2}$.
- $\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt}$ = $2\pi r$ x $\frac{1}{\pi r^2}$ x 0.18.
- Since the radius is 2.5, then $\frac{dA}{dt}$ = $2\pi(2.5)$ x $\frac{1}{\pi(2.5)^2}$ x 0.18 = 0.144 m^2/h.

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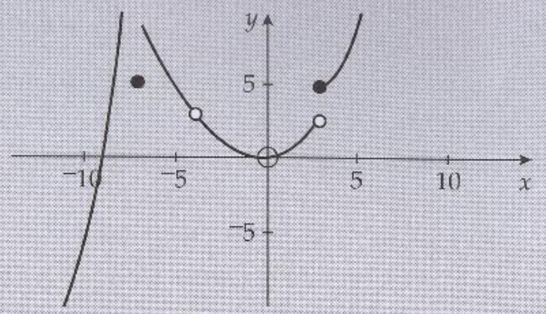
Exercise 3.1C M11 on CD

1 Differentiate $y = 3x^2 - 4x$ from first principles.

2 Draw the graph of the function $y = x^4 - 8x^2 + 16$ and immediately below draw the graph of its derived function.

3



From the graph above, give the values of x for which:

a the function is not defined

b the function is not differentiable

c the function is not continuous.

4 Differentiate the following (you do not need to simplify):

a $y = \dfrac{1 + e^x}{e^{x-2}}$

b $y = 5\ln(2\sin 3x)$

5 Find $\dfrac{dy}{dx}$ of the following:

a $(x-2)^2 + (y-3)^2 = 4$

b $x^4 - 3xy^3 = 9$

6 Find the following limits:

a $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$

b $\lim\limits_{x\to 3} \dfrac{x^2 + 2x - 15}{x^2 - 9}$

7 Sketch the curve with the following properties:

- It is discontinuous at $(0, 0)$.
- It is not differentiable at $(1, 4)$.
- The domain is $^-2 < x < 5$.
- $\lim\limits_{x\to 0} f(x) = 0$.
- It is concave up for $^-2 < x < 1$.

8 The Oncology Department at Dunedin Public Hospital is testing a new medication on a group of 50 patients. It is found that the concentration of the drug in the blood stream after t hours is modelled by the equation:

$$D(t) = \frac{50t}{2t^2 + 1}$$

where D is the amount of drug in the blood measured in parts per million and t is the time in hours after being injected with the dose.

a Calculate $D(0.3)$, $D(0.5)$, $D(1)$, $D(2)$, $D(5)$.

b Calculate $D'(0.5)$, $D'(1)$, $D'(5)$.

c Interpret these results in terms of the amount of drug in the patient's blood.

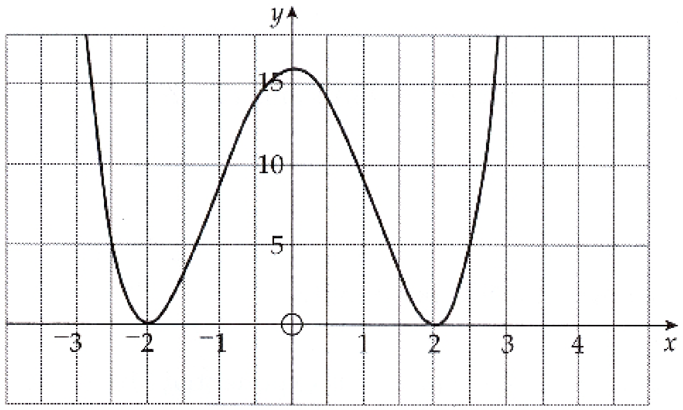
9 An open box is made by removing squares of area x^2 from each corner of a rectangular sheet of cardboard and then folding up the sides. The height of the box is to be half the length of the width. The volume of the box is to be 3600 cm^2. Find the dimensions of the box which will give a minimum surface area (A). You may assume $\dfrac{d^2A}{dx^2} > 0$.

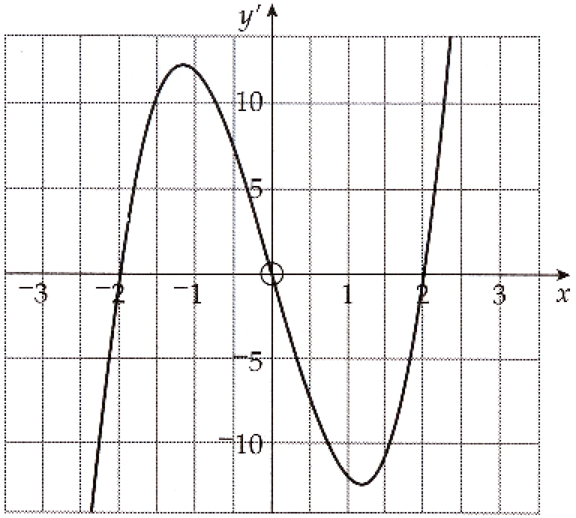
10 A spherical balloon is being inflated with helium from a pump which dispenses helium at the rate of 200 cm^3 per second. At what rate will the radius of the balloon be increasing at the moment when the diameter is 10 cm?

Exercise 3.1C M11

1 $f'(x) = 6x - 4$

2





3 a Function is not defined at $x = {}^{-}4$

b Function is not differentiable at $x = {}^{-}7, {}^{-}4, 3$

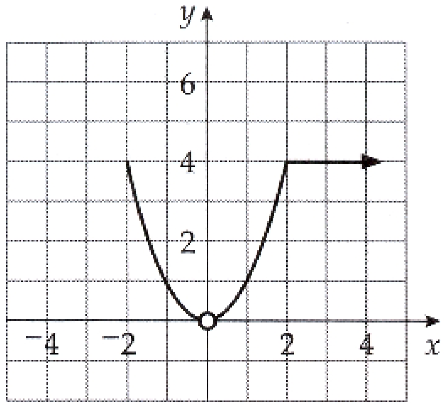
c Function is not continuous at $x = {}^{-}7, {}^{-}4, 3$

4 a $y' = \dfrac{e^{x-2}(e^x) - (1 + e^x)(e^{x-2})}{e^{2(x-2)}}$

b $y' = 5 \times \dfrac{1}{2\sin(3x)} \times 6\cos(3x)$

6 a $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = 1$ b $\lim\limits_{x\to 3} \dfrac{x^2 + 2x - 15}{x^2 - 9} = \dfrac{4}{3}$

7



8 a

t	$D(t)$
0.3	12.71
0.5	16.67
1	16.67
2	11.11
5	4.90

b

t	$D'(t)$
0.5	11.11
1	⁻5.55
5	⁻0.942

c From part a: the body absorbs the drug, the amount in the blood increases to a peak then slowly decreases until very little left after 5 hours (4.90 ppm).

From part b: the rate of uptake of the drug is very high at first, then it becomes negative as the body uses the drug.

9 For minimum surface area, the dimensions need to be:
height = 9.65 cm
width = 19.3 cm
length = 19.3 cm

10 The radius of the balloon is increasing at the rate of $\frac{2}{\pi}$ cm/s (0.637 cm/s) when $r = 5$.